Deflection of Beam

Prepared by: Prof. Rashid Mustafa
Assistant Professor & Head
Department of Civil Engineering
Katihar Engineering College, Katihar

Outlines

- Introduction
- Important Formula
- Double Integration Method
- Macaulay's Method
- Moment Area Method
- Strain Energy Method
- Castigliano's Theorem
- Conjugate Beam Method
- Superposition Theorem
- Maxwell Reciprocal Theorem
- Important Links

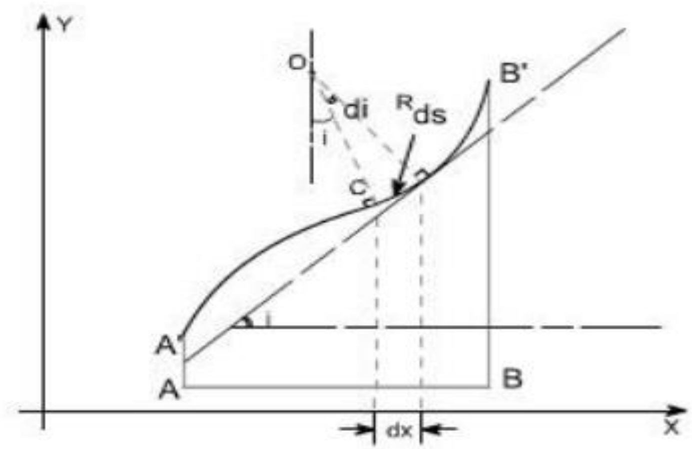
Introduction

- The design of beams is frequently governed by rigidity rather than strength. For example, building codes specify limits on deflections as well as stresses. Excessive deflection of a beam not only is visually disturbing but also may cause damage to other parts of the building. For this reason, building codes limit the maximum deflection of a beam to about 1/360 th of its spans.
- A number of analytical methods are available for determining the deflections of beams. Their common basis is the differential equation that relates the deflection to the bending moment.
- The solution of this equation is complicated because the bending moment is usually a discontinuous function, so that the equations must be integrated in a piecewise fashion

Contd..

- In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.
- In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.
- Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam
- Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

- The curvature is always small.
- Any deflection resulting from the shear deformation of the material or shear stresses is neglected. It can be shown that the deflections due to shear deformations are usually small and hence can be ignored



- Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or in fact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve

- In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram.
- it is assumed that the simple bending theory equation holds good.
- Bending Equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

- To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.
- let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di
- But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

or

$$\boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

Relationship between Shear force, Bending moment and Deflection

- Slope = dy/dx
- Bending Moment(M) = $B.M = EI \frac{d^2y}{dx^2}$
- Shear Force(V) = $\text{Shear force} = EI \frac{d^3y}{dx^3}$
- Load Intensity = $\text{load distribution} = EI \frac{d^4y}{dx^4}$

Standard Results for slope and deflection

Sr. No.	Types of Beam and loading condition	Maximum Slope (ϴ)	Maximum Deflection (δ)
1.	Cantilever beam having point load W at the free end	$WL^2/2EI$	$WL^3/3EI$
2.	Cantilever beam having udl w throughout the span	$wL^3/6EI$	$wL^4/8EI$
3.	Cantilever beam having applied moment M at the free end	ML/EI	$ML^2/2EI$
4.	Simply supported beam having point load W at the centre of the beam	$WL^2/16EI$	$WL^3/48EI$
5.	Simply supported beam having udl w through entire span	$wL^3/24EI$	$5wL^4/384EI$
6.	Fixed beam having point load W at the centre	Slope at supports=0 Slope at the centre =0	¼($WL^3/48EI$)
7.	Fixed beam having udl w throughout of the span	Slope at supports=0 Slope at the centre =0	1/5($5wL^4/384EI$)

Methods to compute slope and deflection

- Double Integration Method
- Macaulay’s Method
- Moment Area Method (Mohr’s Method)
- Strain Energy Method
- Castigliano’s Theorem
- Conjugate Beam Method
- Superposition Method
- Maxwell Reciprocal Theorem

Double Integration Method

- The primary advantage of the double integration method is that it produces the equation for the deflection everywhere along the beams.
- If concentrated point load and concentrated moment is acting then bending moment equation continuously changes from move left to right or right to left in the span of the beam, then this method is not useful.
- **Direct integration method:** The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

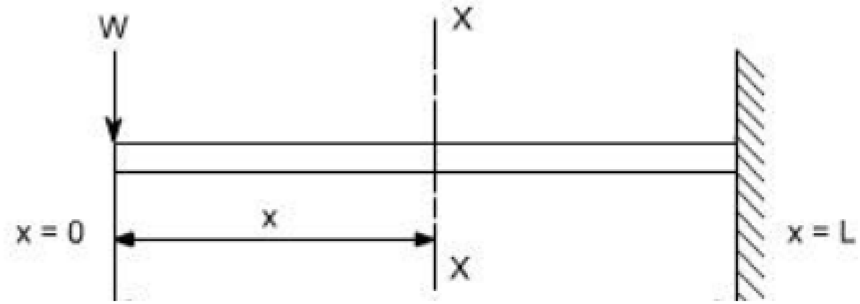
$$\frac{dy}{dx} = \int \frac{M}{EI}\,dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}\,dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

- Problem:1 Cantilever Beam with concentrated load at the free end of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}\,dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}\,dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

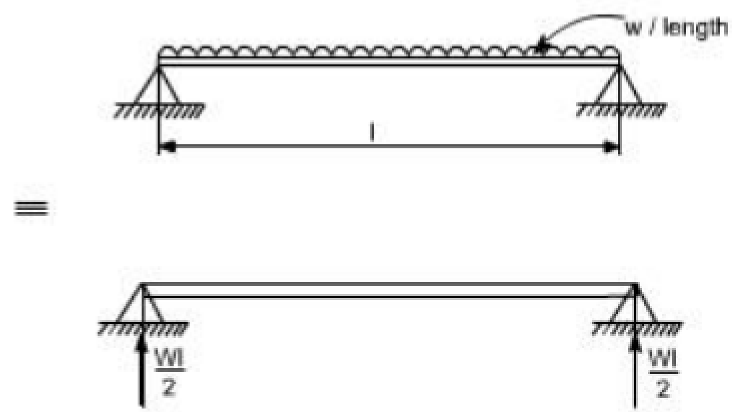
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

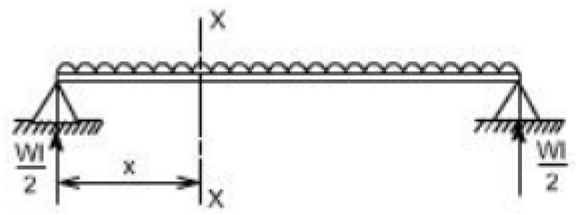
$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

Problem:2 A simply supported beam having uniformly distributed load throughout of the beam



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



Contd…

$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad ----(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

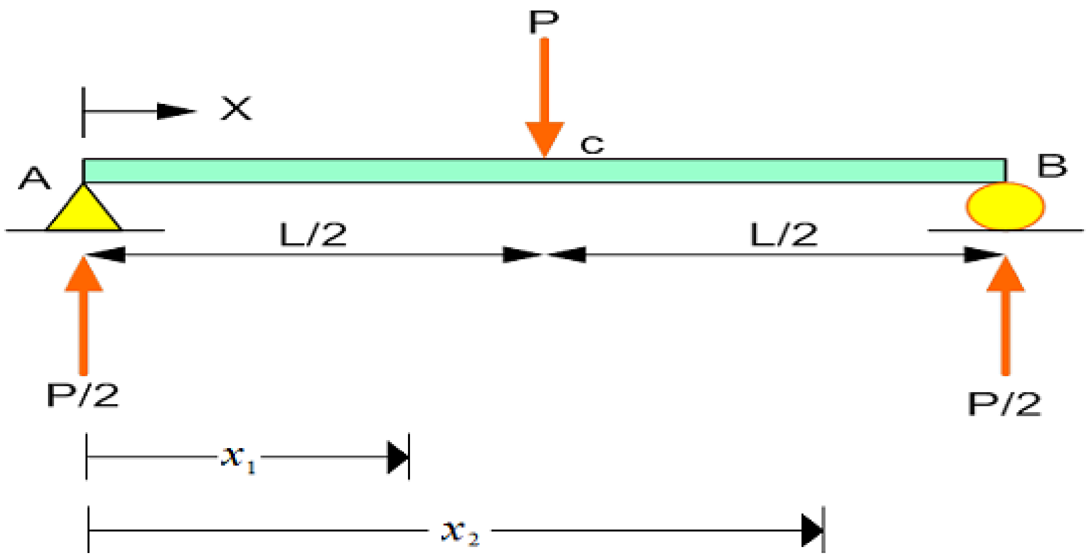
Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Macaulay’s Method

- Useful when equation of bending moment changes from one part to other part of span such as beam contained concentrated point load and concentrated moment load.
- Macaulay’s method is an improvement over double integration method.
- Macaulay’s method is a means to find the equation that describes the deflected shape of a beam
- From this equation, any deflection of interest can be found
- Macaulay’s method enables us to write a single equation for bending moment for the full length of the beam
- When coupled with the Euler-Bernoulli theory, we can then integrate the expression for bending moment to find the equation for deflection using the double integration method.
- Macauly’s Method allow us to ‘turn off’ partial of moment function when the value inside a bracket in that function is zero or negative

Let us again consider a simply supported beam AB of length L and carrying concentrated load P at mid span,C as shown below. EI is constant. This example are going to show how to find the equation of elastic curve for the beam by 'turn off' part of a function using Macauly's Method.



- Again we must write a function for the beam moment that can describe the moment for the beam wholly from the left side.
- This beam have 2 span. Macauly's Method will use the moment function to the very right with only x function as distance. Where here for example:
- Span $L/2<x<L$

$$M = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

- Take note here Macauly's Method use a different bracket that have a special function that have an advanced understanding and application.

- Span L/2<x<L

$$M = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

- The bracket above allow the function of 'turn off' when inside value is negative or zero.
- Means if we have $x \le \frac{L}{2}$ the $\left\langle x - \frac{L}{2}\right\rangle$ will be zero
- Mathematically explained as :

$$\left\langle x - \frac{L}{2}\right\rangle = \begin{cases} 0 \quad , & x \le \frac{L}{2} \\ x - \frac{L}{2}, & x > \frac{L}{2} \end{cases}$$

- Applying Euler-Bernoulli Theory replace M into slope and displacement integration.

$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle$$

$$\theta EI = \frac{dv}{dx} = \int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$vEI = f(x) = \int\int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

- From the slope integration :

$$\theta EI = \int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$\theta EI = \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 + C_1$$

- Take note here that $\left\langle x - \frac{L}{2}\right\rangle$ is integrate as a function of x. This is rooted to advanced math that Macaulay use in his method that need to be remember.

- From the displacement integration :

$$vEI = \int\int \frac{P}{2}x - P\left\langle x - \frac{L}{2}\right\rangle dx$$

$$vEI = \int \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 + C_1\, dx$$

$$vEI = \frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 + C_1 x + C_2$$

- Again Ttke note here that $\left\langle x - \frac{L}{2}\right\rangle$ is integrate as a function of x.
- From slope and displacement integration procedure, 2 unknown were obtained and solved using the boundary condition:
  - $v = 0, x = 0$
  - $v = 0, x = L$
- Please remember :

$$\left\langle x - \frac{L}{2}\right\rangle = \begin{cases} 0 \quad , & x \leq \frac{L}{2} \\ x - \frac{L}{2}, & x > \frac{L}{2} \end{cases}$$

- Lets use the first boundary
  - $v = 0, x = 0$

$$0EI = \frac{P}{12}0^3 - \frac{P}{6}\left\langle 0 - \frac{L}{2}\right\rangle^3 + C_1 0 + C_2$$

- Inside the bracket $\left\langle x - \frac{L}{2}\right\rangle = -\frac{L}{2} = 0$

$$0EI = \frac{P}{12}0^3 - 0 + C_1 0 + C_2$$

$$C_2 = 0$$

- The second boundary
  - $v = 0, x =$ L and $C_2 = 0$

$$0EI = \frac{P}{12}L^3 - \frac{P}{6}\left\langle L - \frac{L}{2}\right\rangle^3 + C_1 L$$

- Inside the bracket $\left\langle L - \frac{L}{2}\right\rangle = \frac{L}{2}$ we use the value

$$C_1 = -\frac{3PL^2}{48}$$

– $C_1 = -\frac{3PL^2}{48}$ and $C_2 = 0$

$$\theta EI = \frac{P}{4}x^2 - \frac{P}{2}\left\langle x - \frac{L}{2}\right\rangle^2 - \frac{3PL^2}{48}$$

$$vEI = \frac{P}{12}x^3 - \frac{P}{6}\left\langle x - \frac{L}{2}\right\rangle^3 - \frac{3PL^2}{48}x$$

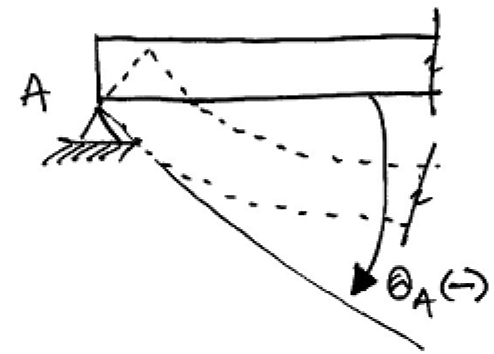
- Lets determine slope at the support
  - At $x = 0$

$$\theta EI = \frac{P}{4}0^2 - \frac{P}{2}\left\langle 0 - \frac{L}{2}\right\rangle^2 - \frac{3PL^2}{48}$$

- Inside the bracket $\left\langle x - \frac{L}{2} \right\rangle = -\frac{L}{2} = 0$

$$\theta EI = -\frac{3PL^2}{48}$$

$$\theta = -\frac{PL^2}{16EI}$$



At $x = L$

$$\theta EI = \frac{P}{4}L^2 - \frac{P}{2}\left\langle L - \frac{L}{2} \right\rangle^2 - \frac{3PL^2}{48}$$

Inside the bracket $\left\langle x - \frac{L}{2} \right\rangle = \frac{L}{2}$

$$\theta EI = \frac{3PL^2}{48}$$

$$\theta = +\frac{PL^2}{16EI}$$

- Lets determine maximum displacement at the midspan :
  - At $x = \frac{L}{2}$

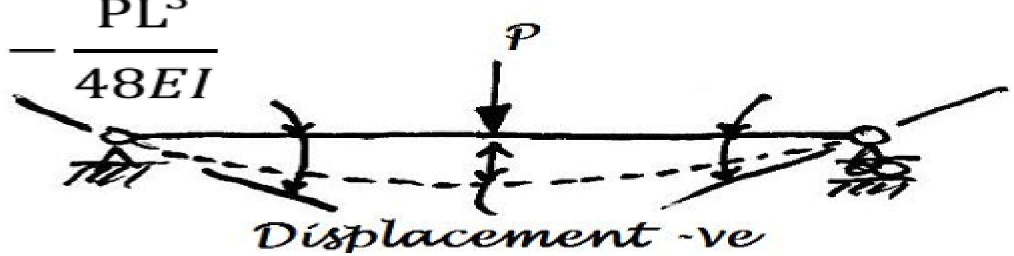
$$vEI = \frac{P}{12}\left(\frac{L}{2}\right)^3 - \frac{P}{6}\left\langle \frac{L}{2} - \frac{L}{2} \right\rangle^3 - \frac{3PL^2}{48}\left(\frac{L}{2}\right)$$

- Inside the bracket $\left\langle x - \frac{L}{2} \right\rangle = 0$

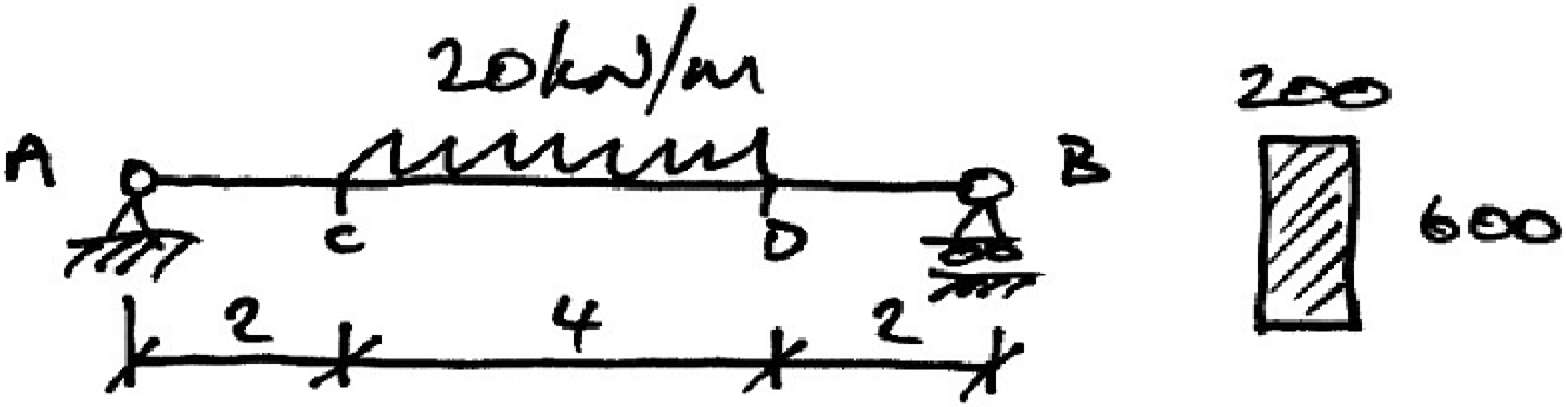
$$vEI = -\frac{PL^3}{48}$$

$$v = -\frac{PL^3}{48EI}$$

- Negative means downward

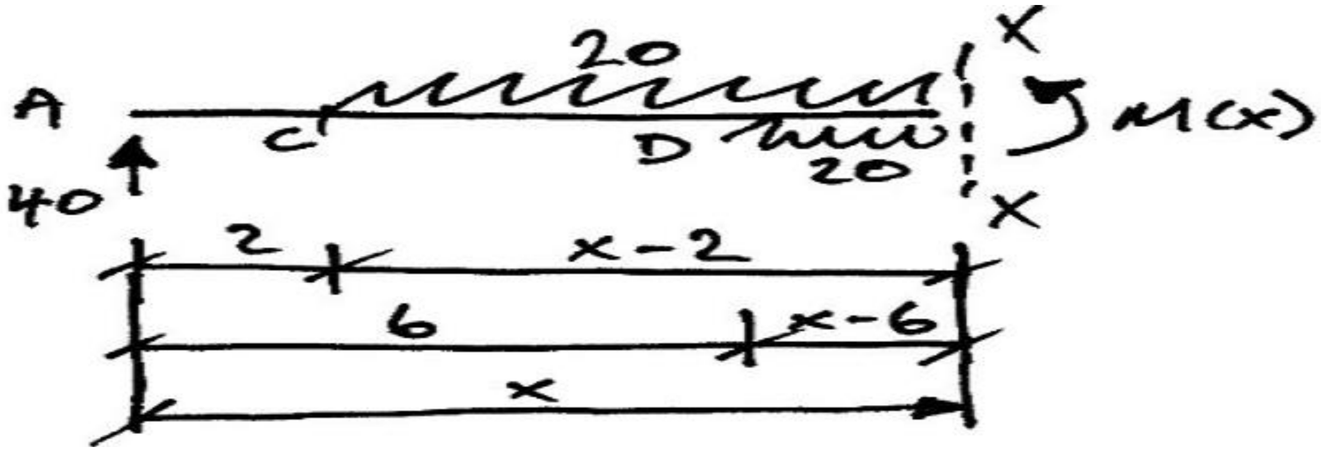


Problem: A simply supported beam having udl of 20 kN/m applied as shown in figure. Take E = 30 kN/mm2



$$I = \frac{bd^3}{12} = \frac{200 \cdot 600^3}{12} = 36 \times 10^8 \text{ mm}^4$$

$$EI = \frac{(30)(36 \times 10^8)}{10^6} = 108 \times 10^3 \text{ kNm}^2$$



$$M(x) - 40x + \frac{20}{2}[x-2]^2 - \frac{20}{2}[x-6]^2 = 0$$

$$M(x) = 40x - \frac{20}{2}[x-2]^2 + \frac{20}{2}[x-6]^2$$

Applying Euler-Bernoulli (v = y):

$$M(x) = EI\frac{d^2y}{dx^2} = 40x - \frac{20}{2}[x-2]^2 + \frac{20}{2}[x-6]^2 \qquad (1)$$

Integrate Equation 1 to get the slope equation

$$EI\frac{dy}{dx} = \frac{40}{2}x^2 - \frac{20}{6}[x-2]^3 + \frac{20}{6}[x-6]^3 + C_\theta \qquad (2)$$

Integrate Equation 2 to get the displacement equation

$$EIy = \frac{40}{6}x^3 - \frac{20}{24}[x-2]^4 + \frac{20}{24}[x-6]^4 + C_\theta x + C_\delta \qquad \textbf{Equation 3}$$

The boundary conditions are:

– Support A: y =0 at x = 0

– Support B: y =0 at x = 8

So for the first boundary condition:

$$EI(0)=\frac{40}{6}(0)^3-\frac{20}{24}[0-2]^4+\frac{20}{24}[0-6]^4+C_\theta(0)+C_\delta$$

$$C_\delta=0$$

For the second boundary condition:

$$EI(0)=\frac{40}{6}(8)^3-\frac{20}{24}(6)^4+\frac{20}{24}(2)^4+8C_\theta$$
$$C_\theta=-293.33$$

Insert constants into Equations 3 (displacement)

$$EIy=\frac{40}{6}x^3-\frac{20}{24}[x-2]^4+\frac{20}{24}[x-6]^4-293.33x$$

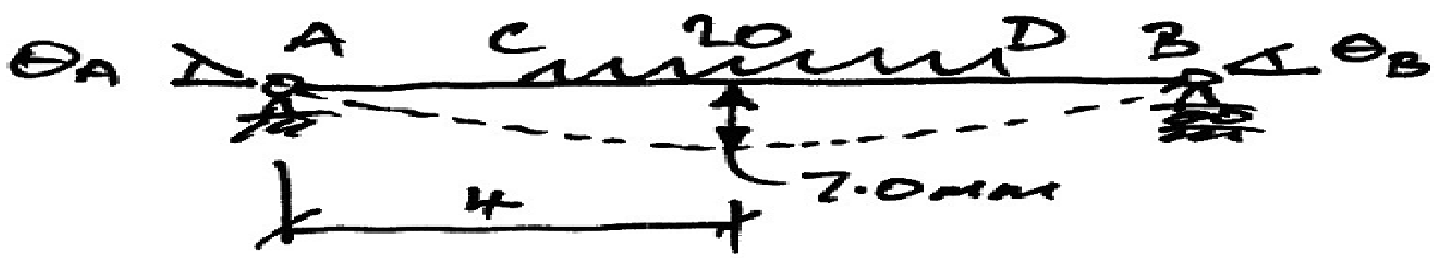
$$EI\delta_{max} = \frac{40}{6}(4)^3 - \frac{20}{24}(2)^4 + \cancel{\frac{20}{24}[4-6]^4} - 293.33(4)$$

$$= -760$$

$$\delta_{max} = \frac{-760}{EI} = \frac{-760}{108 \times 20^3} = -0.00704 \text{ m}$$

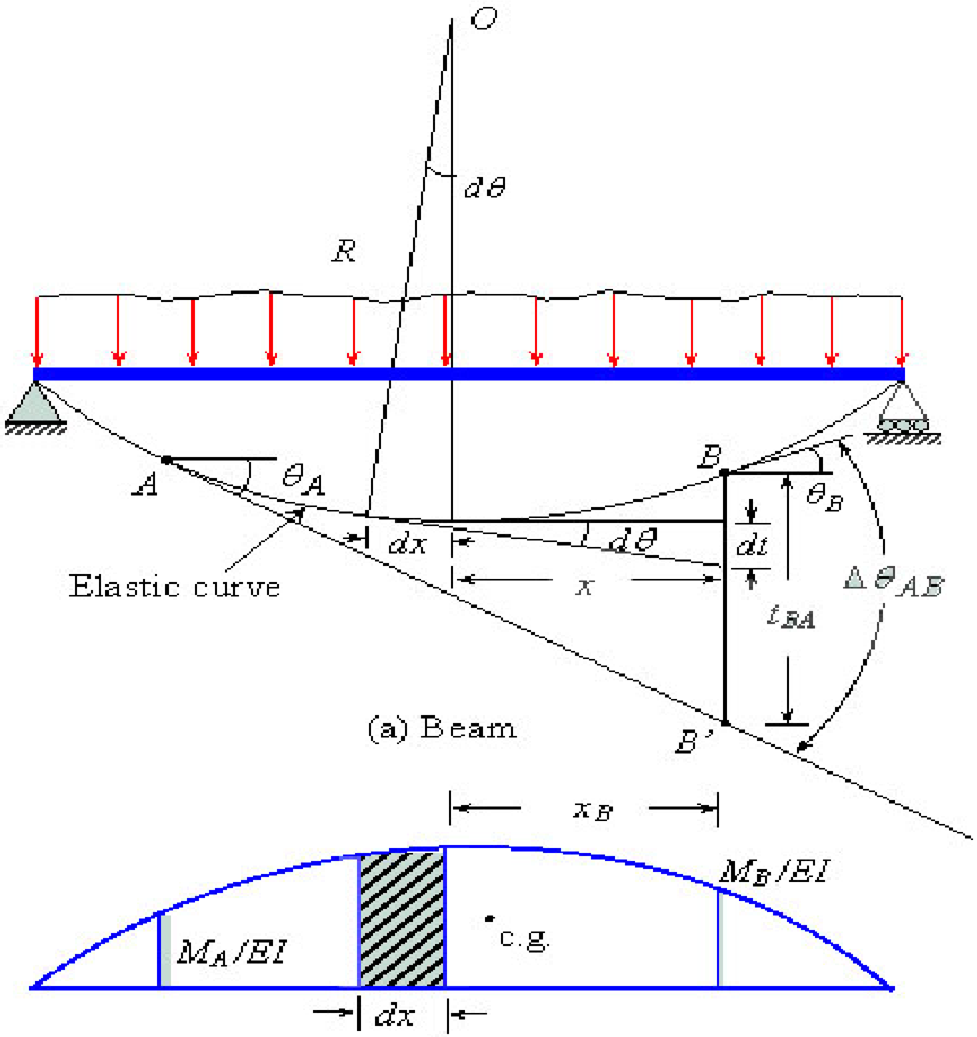
$$\delta_{max} = -7.04 \text{ mm}$$

This is therefore a downward deflection as expected.



Moment Area Method

- Effective methods for obtaining the bending displacement in beams and frames
- In this method, the area of the bending moment diagrams is utilized for computing the slope and or deflections at particular points along the axis of the beam or frame.
- Two theorems known as the moment area theorems are utilized for calculation of the deflection.
- One theorem is used to calculate the change in the slope between two points on the elastic curve.
- The other theorem is used to compute the vertical distance (called tangential deviation) between a point on the elastic curve and a line tangent to the elastic curve at a second point.



(b) M/EI diagram

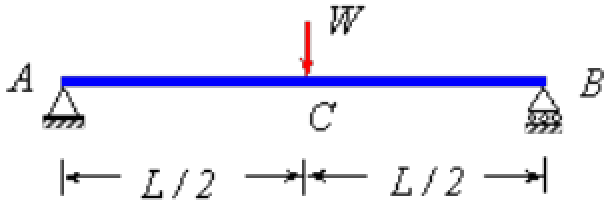
$$\theta_B - \theta_A = \text{Area of } M / EI \text{ diagram between } A \text{ and } B \qquad \text{------------(1)}$$

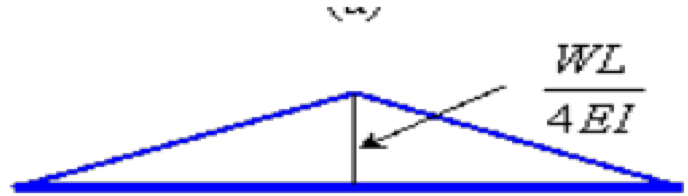
Equation 1 is also known as Mohr's First Theorem

$$t_{BA} = \int_A^B x\,d\theta = \int_A^B \frac{Mx}{EI}\,dx \qquad \text{-------(2)}$$

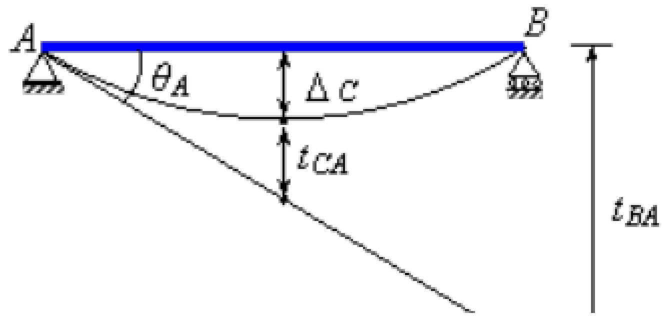
Equation 2 is also known as Mohr's second theorem

Problem: Determine the end slope and deflection at mid point C in the beam shown in figure below using moment area method





M/EI diagram



$$\theta_A \times L = t_{BA}$$

$$\theta_A = \frac{1}{L} \times \left(\frac{1}{2} \times \frac{WL}{4EI} \times L \times \frac{L}{2} \right) = \frac{WL^2}{16EI} \text{ (clockwise direction)}$$

The slope at B can be obtained by using the first moment area theorem between points A and B i.e.

$$\theta_B - \theta_A = \Delta\theta_{AB}$$

$$\theta_B - \theta_A = \frac{1}{2} \times \frac{WL}{4EI} \times L = \frac{WL^2}{8EI}$$

$$\theta_B = \frac{WL^2}{8EI} - \frac{WL^2}{16EI} = \frac{WL^2}{16EI} \text{ (anti-clockwise)}$$

(It is to be noted that the $\theta_A = -\frac{WL^2}{16EI}$. The negative sign is because of the slope being in the clockwise direction. As per sign convention a positive slope is in the anti-clockwise direction)

The deflection at the centre of the beam can be obtained with the help of the second moment area theorem between points A and C i.e.

$$\theta_A \times \frac{L}{2} = \Delta_C + t_{CA}$$

$$\frac{WL^2}{16EI} \times \frac{L}{2} = \Delta_C + \left(\frac{1}{2} \times \frac{WL}{4EI} \times \frac{L}{2} \times \frac{L}{6}\right)$$

$$\Delta_C = \frac{WL^3}{48EI} \text{ (downward direction)}$$

Important Links

- https://nptel.ac.in/content/storage2/courses/105101085/downloads/lec-22.pdf
- https://nptel.ac.in/
- https://talktorashid.blogspot.com/
- https://www.youtube.com/watch?v=QhUmY_zyI3o
- https://www.youtube.com/watch?v=qzpLRjKSGb0
- http://keck.ac.in/department/civil/rm


THANK
YOU